An Equation Mis-Pelled

Uncle Bob

"A person who can within a year solve the equation

$$x^2 - 92y^2 = 1$$

is a mathematician." (Brahmagupta, c.600 A.D.)

I happened upon this bit of Eastern provocation while reading a chapter on Diophantine equations, an ancient topic in number theory. These are named for the Greek Diophantus (c.300 A.D.) who was an important bridge from classically geometric approaches to a more algebraic- and number-centered approach to mathematics.

A very simple Diophantine example is the equation $x + y = 7$, which asks for two numbers that add to seven. Seems too simple, but ask first what type of numbers x and y can be. If we are looking for just positive whole solutions there are only six. Can you list them? If we allow the negative integers, however, there exists an infinitude of solutions, (-1, 8), (-2, 9), (-3, 10) and so on. Diophantine problems can have two or more unknowns and utilize the second, third, and higher powers of variables as well. They are also referred to as indeterminate problems because, on the surface, one cannot tell if there are any, several, or countless solutions. And on the surface, pretty dry stuff, eh?

To whet your appetite for this matter, let me quote an ancient problem which is modeled by such equations. We approach a forest whose numerous trees have their branches "bent down with the weight of flowers and fruits, trees such as jambu, date palms [etc.], and mango trees – filled with the many sounds of parrots and cuckoos found near springs containing lotuses with bees... a number of travelers entered with joy." There were 63 heaps of equal numbers of fruit and a separate pile of seven fruits. These were all divided equally among 23 travelers. Tell the number of fruits in each heap [Ore, Number Theory and Its History]. There – that's a pretty juicy problem!

Brahmagupta's equation at the top is another example. It belongs to a family known as Pell's equations. In the general Pell equation, the 92 can be any non-square number. Pell lived in the time of, and in service of, Cromwell's England. He did make contributions to number theory, including a book listing all the factors of numbers less than 100,000, but someone (turns out it was the great Euler) got it wrong in crediting solutions of these equations to him. They were mis–Pelled.

More likely the credit for progress in that era belongs to Brouncker and Wallis, two Englishmen who were provoked by challenges from Pierre Fermat of "Last Theorem" fame. Fermat cannot be given credit for initiating the study of numbers, and we will see further evidence of that. He can be pointed to as the European who re-kindled it from antiquity's ashes, inspired others to study it, and brought it to the status of true science. His Last Theorem perplexed many great minds for over three hundred years and was finally proved by Wiles in 1994. It involves a Diophantine problem.

You see, Pythagoras' Theorem shows that certain square numbers can be separated into two square parts, such as 25 into 9 and 16. (Can all squares be so separated?) Fermat claimed that a cube (cubic number) could never be separated into two cubes, and that there were no like solutions in any higher power either. He scribbled this idea in the margin of a page containing a Pythagorean example, in a translation of an arithmetic book written by none other than Diophantus.

Well super, if you like numbers and equations, but I hoped that, if I looked deeper into the history of these equations, I would find a few good stories, and I did. C.D. Olds in his book on continued fractions alludes to a "cattle problem" given by Archimedes to his friend and colleague Eratosthenes. Olds describes it as having eight unknown quantities, but he also says it reduces to solving Pell's equation with the 92 replaced by 4,729,494!! The smallest numbers that solve this Pell are 45 and 41 digits long. When this is worked back into the cattle problem, which is also known as the "Rancher's Dilemma," the solutions involve numbers that are hundreds of thousands of digits long.

What follows is a summary of a translation I found in Wilson's The Infinite in the Finite. The problem exhorts "the stranger" to compute the number of cattle of the Sun. There are cows and bulls of four different breeds and in various specific proportions to one another. If the stranger can tell the numbers of bulls and cows, s/he would not be counted as unskilled or ignorant, but would not yet be counted among the wise. You see, in addition, the sum of two of the breeds must be a square number, and the rest must be a triangular number, and the totals must be broken down by breeds and genders, so this ups the stakes considerably. The drudge work was not completed until 1880 by Amthor, and it is debatable whether this was wise on his part. The solution alone fills 80 pages.

Finally, I cannot fail to report that the [mis-Pelled] equation was solved 500 years before Pell, and in the Orient. Bhaskara of Ujjain in the Indian subcontinent, writing in the 12th century, describes a 'cyclic method' which zeros in on the least solution to a Pell equation. It has a trial and error flavor, but it is essentially like the method formalized later by the Europeans. Bhaskara's book, the Lilavati, is named for his daughter. Wilson relates the story of this dedication. The daughter had received an omen that she would never marry. Her father placed an "hour cup" in a pool. The cup had a small hole that slowly let in water. The cup was known to sink in one hour's time, and if it did so again, he told his daughter, she could ignore the omen. In leaning over to watch the cup, however, she caused a tiny pearl to fall from her sari into the cup, thus blocking the hole and keeping the cup afloat. Bhaskara consoled his daughter by giving her name to his book and honoring her in saying, "...for a good name is a second life and the ground work of eternal existence."

And to bring these tales full circle, I mention that an important part of Bhaskara's cyclic method is called the 'pulverizing' step. It is too detailed to include here, but I can tell you that it was used for calculating the orbits of the planets in the early 7th century by another Ujjainian: the man who, for Bhaskara, was, without doubt, a source of information, inspiration, and perhaps provocation. He was named Brahmagupta.

Source:
Alistair Wilson. *The Infinite in the Finite*. Oxford, 1995.

Your problem is entitled **The Consumer's Dilemma**, and I hope you enjoy working on it: The Lackwicks, Jim and Karen, are very excited about their $600 tax rebate check which is due in the mail any day. They plan to spend the full amount at heretoday.com, their favorite online shopping website. Heretoday is offering a Rebate Blowout Sale, and the only restriction is a one hundred item limit for the shopping cart. Jim and Karen were going to purchase three things in quantity (at least one of each): Carrot-Patch Kids for $18 each; Bubble Bunnies for $12 each; and Gummy Hares for $3 apiece. Describe all the ways in which the Lackwicks can purchase a total of 100 of these items for exactly six hundred dollars.

Solution on page 3

Solution

The Consumer's Dilemma

One solution is 8, 20, and 72 Carrot Patch Kids, Bubble Bunnies and Gummy Hares, respectively. The number of Gummy Hares must be even because of the sole odd unit price.

Bonus: Dropping 3 Kids and 2 Gummy Hares, we can pick up 5 Bunnies and keep the totals at 100 items and $600. Or we can add 3 Kids and 2 Hares and drop 5 Bunnies. Can you find five other solutions?

See more solutions below.

Carrot Patch Kids	Bubble Bunnies	Gummy Hares
2	30	68
5	25	70
8	20	72
11	15	74
14	10	76
17	5	78

That's All, Folks!