Threading Through Math History

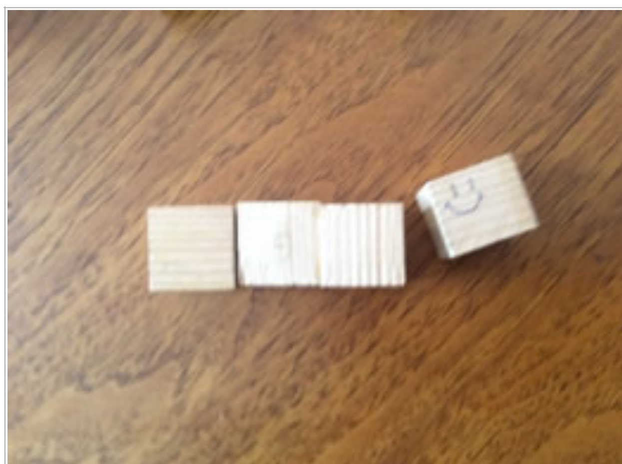
Infinity – and Then Some!

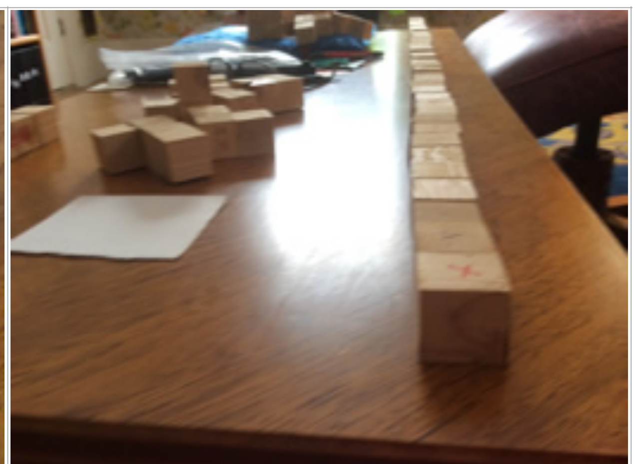
Uncle Bob

Let's say I have a ball of clay the size of a golf ball. After softening it, I roll it into one of those worms the school kids like to make. The worm might be seven or eight inches long, but it contains the same amount of clay. If I start with a baseball's worth of clay, I might be able to roll a worm three feet in length.

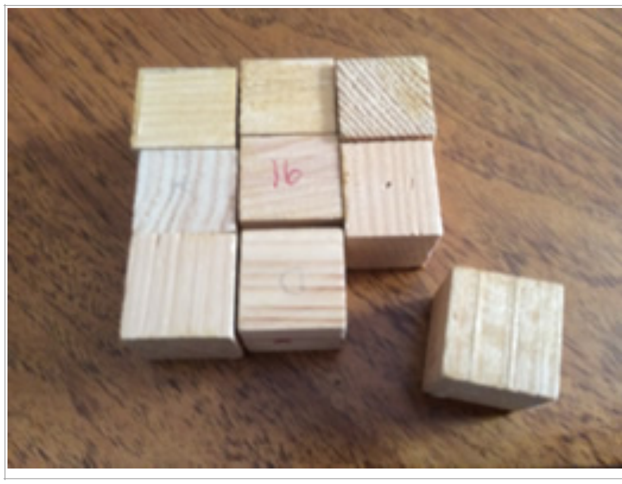
I can reverse this process too. The worms can be reshaped to their original spheres without many clay molecules lost on my hands. Now a question for you: if I had a worm infinite in length, and I reshaped it into a sphere, what would I have? We'll return to this question.

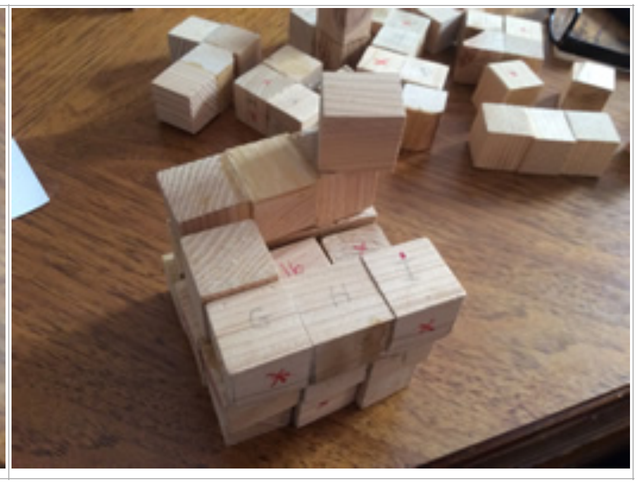
The 14th century clergyman and philosopher Albert of Saxony, Bishop of Halberstadt and first rector of the University of Vienna, was fond of doing thought experiments, some of which dealt with infinite concepts. His thoughts strike me as very progressive for the pre-Renaissance era that still clung to Aristotle's world view — and an abhorrence of the idea that infinity could be anything but hocus-pocus. One of Albert's questions put infinity in concrete, or at least wooden terms.





His question centered around a wooden beam of infinite length. The way he imagined it sawn and reassembled might amaze you. What if Albert's infinite supply of blocks were wrapped infinitely around one block; would there be enough for an infinite floor? What if, in three dimensions, he surrounded that central block with his infinite supply? What would be created?





[Read Albert's story here.]

Consideration of the infinite became essential in the development and refinement of the calculus in the 17th through the 19th centuries. Could we even contemplate slicing up a region into finer and finer pieces until the number of them approached infinity, and then add them up and expect a meaningful result? That's what Newton and Leibniz expected of their integration techniques. It had worked in some cases for Archimedes 2000 years earlier.

First, to the meaningful question. Can adding an infinite number of quantities produce a number we can understand? Well clearly, adding an infinite number of ones, that is

1 + 1 + 1 + 1 + …,

gives us an unlimited sum. Contrast that result with my hike to the store. The store is one mile away. On the first day of the hike I go half a mile, get tired, set up a little personal campsite beside the road, have a bite to eat, and go to sleep. On the second day, I'm even less ambitious and I cover only half of the remaining distance, or one-quarter mile. Now my thought experiment asks, what if I cover only half of the remaining distance on each successive day. Ridiculous, yes, but if I quit walking after any number of days, even a thousand days, I will not have reached the store. The full trip will require a time greater than any number of days, and yet the sum of all the legs of my journey add to not more than a mile. I could imagine that an infinite number of days gets me to the store, and the calculus of integration gets you there too.

Now, as to Archimedes, he did much the same summing, but in a more concrete manner. His areas he imagined as laminas — plates of negligible thickness. To get the area of some irregular shape he would slice it into lamina and reshape them into a region of calculable area. Imagine a bookshelf of a set of encyclopedias all leaning at a severe angle. What is the total area of all the spines? Well, we can straighten them up and have something closer to a rectangle of book spines, and then we will know how to figure it.

In the late 1800s, Georg Cantor revived the thought experiments on infinite sets. He assumed that the counting set {1, 2, 3, 4, 5, …} had no largest number. He reasoned that to count other sets he could match those elements up with the counting numbers, and he very quickly saw some strange results. He counted the even numbers and found they were just as big a set. After all, every counting number has a double, and every even number has a half which is a counting number.

You might be thinking, "Wait. The even numbers are only half of the full set." That is what most people think at first. There is a complete presentation of the theory here.

From this reasoning we can understand Cantor's thinking. "Two sets might not be equal, in the sense that they contain different elements, but they still might be equal in size, having the same number of elements."

Now we back up to Albert's wooden blocks. He imagined an infinitely long row of them. They clearly do not take up all of space because there is plenty of "space" around them. But if we start building that surrounding box, we can make it larger and larger — we have an infinite supply of blocks — and, in theory, that box, or my infinite ball of clay will take up all of space.

Cantor concluded, "An infinite set is precisely any set that can be placed in 1-to-1 correspondence with a proper subset of itself." The even numbers are a proper subset of the counting numbers, but those two sets match up one-to-one. Cantor found other unlikely pairings in this manner. He also found that certain sets didn't match. The numbers on the real number line, including the irrationals, could not be "counted." That set seemed to be larger — larger than infinity. Suddenly there was a need for more than one way to name infinities.

It was in the attempt to rank them that Cantor made a guess about how some of these sets compared. Much later, after Gödel showed that you couldn't prove everything, Cantor's guess, called the Continuum Hypotheses, was shown to be not provable one way or t'other. See here.