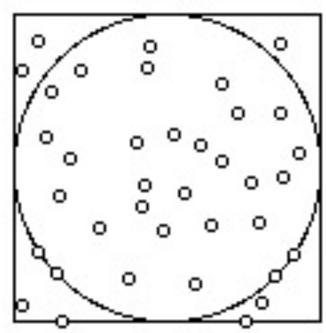
Geometry Probability
The Monte Carlo Method: Can You Close the Deal?
Uncle Bob

First off, we will assure you that your assets are not at risk here. The Monte Carlo method does involve trials and probabilities, but it does not involve gambling — unless you want it to. We will use this method to gather some statistics that will estimate some geometric quantities for us.

As an **example**, let's say that we remember from high school that the area of a circle is, um, "something" times the radius squared. We will collect some data on a virtual dartboard to try to estimate that "something." We shoot virtual darts at a square board on which a circle is drawn, as in the diagram below.

If we say that the side of the board is 2, then the radius of the circle, and the radius squared are both one. So the area of the circle is that 'something' we are looking for times one. The area of the board is 2 by 2 or 4. If we shoot a million virtual darts, smothering the entire board, and calculate the fraction of the darts that also land inside the circle, multiply that fraction by 4, we should have a good estimate of the 'something.'



Random points in the Unit Square. 78% also in the Unit Circle.

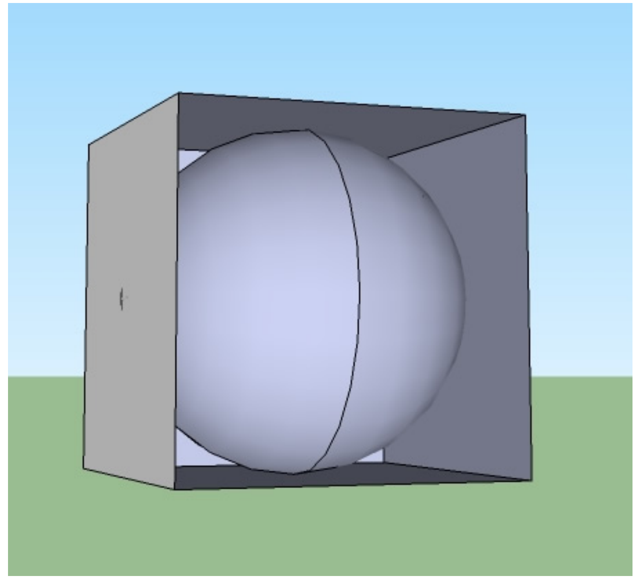
Closing the Deal? The experiment has been performed, and we found that 78.5242% of the darts landed inside the circle. Find that percent of 4 and you will have estimated pi.

4 * 0.785242 = 3.14077. Not too shabby.

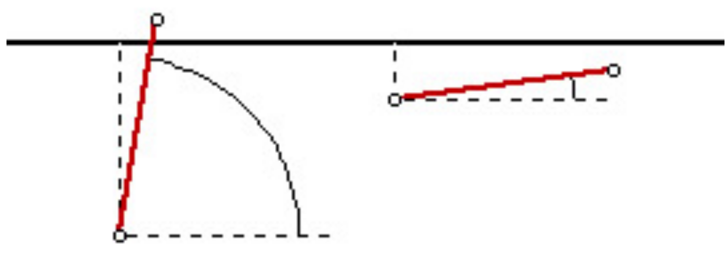
We provide you with three more deals to close. [Solutions at the end of the article]

Case 1. Your first case will be to analyze the three-dimensional version of the example above. We would like an estimate of the volume of a sphere inside a box (think packaging for a basketball), and we want to know if the formula is pi times the radius cubed (to the third power), or something like that.

We can shoot virtual darts in three dimensions too. Computers are great for stuff like that. If the cube has a side measuring 2, then its volume is 8 (2x2x2, right?). This time we find that only 52.3464% of darts land inside the sphere. *Can you close the deal?* Find an estimate for the volume. Is it some multiple of pi? Recall that the radius is again one.



Case 2. Now you investigate a problem called Buffon's needle, named after the 18th century European count. We have a plank floor with boards that are 2″ wide. We drop a 2-inch long needle onto the floor — many, many times. Some will land and cross a crack between planks. Others will land wholly within one board on the floor.



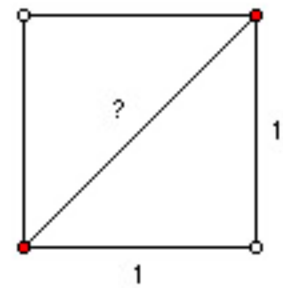
The data is in: yes, another million virtual results. We'll say a few words about how the experiment was performed. You can see from the diagram that whether a needle crosses a crack depends on two things — how close an end is to a crack, and the angle the needle makes with the planking. We used a random location for one end of the needle and a random angle for its orientation, anywhere from pointing straight down to straight up. *Can you close this deal?*

Our theorists are certain that this result involves the number pi also. Yi! Pi again! We found that 63.6786% of the needles landed across a crack. Make this percent into a decimal, as we have done in other examples, and "divide 2 by that decimal."

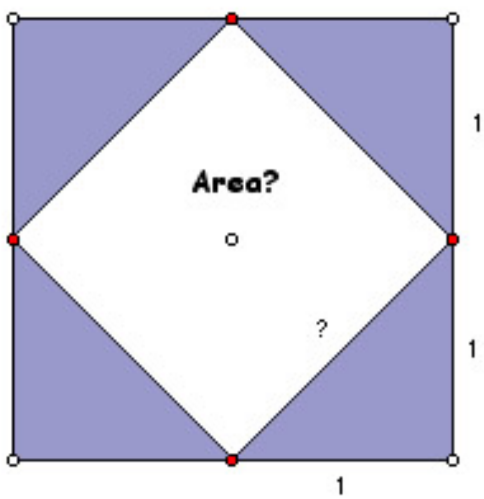
2 / (our decimal) = ??

Case 3. Your last case for now. Forget anything that you do remember from your geometry experience. We will gain experience on two historic questions by experimenting and using statistics. The first question was answered by the Pythagoreans in the 6th century BCE If a square has side measure of one, how long is the diagonal?

The second question was posed by Socrates to a slave in the fifth century BCE The story is recounted in the dialogue "Meno" written by Plato. Given the square above, what would be the side measure of a square double in area? The slave guesses 2 at first, but 2x2 makes a square four times as large.



We answer both questions with a Monte Carlo technique performed on 4 copies of the original, arranged so that there is a square inside a square of area 4.



Can you close the deal? Even if we think or know that smaller square is half of the area 4, we will gather the data, get an estimate of that area, and then take a square root to see how long a side is. Recall that one of those sides is the diagonal of the 1 by 1 square in the first question.

Sure enough, out of one million trials, 50.0228% hit inside the the smaller square. Now find your area estimate and take the square root to approximate the square root of 2. Compare your approximation with your calculator's decimal for the root of 2.

Uncle Bob hopes you will take a gamble and try these. By the way, shooting a million virtual darts using Mathematica® and an iMac® took less than 30 seconds.

Monte Carlo Solutions

Closing the Deal (all figures are rounded)

Case 1. The volume of the sphere is 4.1877 and that's pi times 1.333. (4/3)

Case 2. 2 / 0.636786 = 3.14077 whereas the actual pi is about 3.14159.

Case 3. The area estimate is 2.00091 (in theory it's 2), and the square root of the estimate is 1.41454. Compare this to the calculated $\sqrt{2}$ = 1.41421.