# The Musical Circle of Fifths
A Math Application
"Uncle Bob" Mead

Let's say the clock reads 12 midnight. Jumping ahead in four-hour intervals, the clock will read 4, 8, 12 again, 4 again, 8, 12, and so on. We call it "clock arithmetic" or more prettily "addition modulo 12." If we added 4s beginning with 12 in straight arithmetic, we would get 12, 16, 20, 24, 28, etc., but the clock ignores the extra 12s. Likewise, modular arithmetic looks at only the remainders, and in modulo-12, after ignoring 12s, the remainders are 0, 4, 8, 0, 4, 8, and so on.

The diagram shows a musical clock. Because the keyboard (chromatic) scale contains twelve distinct tones before repeating them in octaves higher or lower, we can situate the notes at the hour marks of a clock face. With C at the top mark, we place all the half-tones running clockwise (imagine that) to C#, D, D#, and so on to note B before returning to a C one octave higher. Note that C-sharp (C#) and D-flat are the same note, but may be spelled one way or the other in different songs. Octave? Doesn't that imply a group of eight? The explanation is ahead.
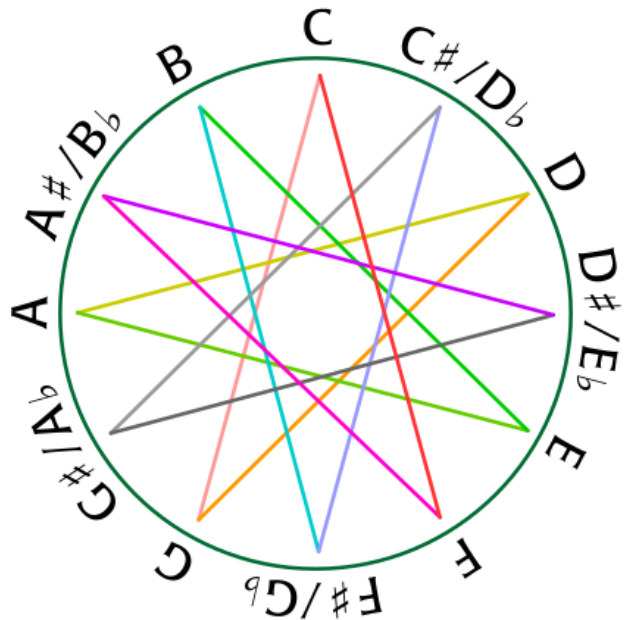
https://en.wikipedia.org/wiki/Circle_of_fifths

Intervals: Most songs do no use all 12 notes in the scale, and melodies can take any number of jumps to the next note. "Twinkle, Twinkle Little Star," in the key of C, takes a leap immediately from C to G. That interval is called a fifth. On the diagram, you can trace a pale coral (?) line from C to G, or from 12 to 7 o'clock. Around the clock, the fifth interval advances seven hours, or seven half-tones.

Other intervals include the major third. You hear this interval in the tunes for "When the Saints Go Marching In" and "Morning Has Broken." From C this interval jumps to E, or four hours, just as 12 advanced to 4 o'clock above. From E we could leap another major third, or four more hours, to G# (G-sharp or 8 o'clock). At the end, I include other intervals and well-known tunes that use them.

Interval Circles: If we ascend from G# another major third (four hours), we arrive back at C. So our arithmetic sequence above, 0, 4, 8, 0, 4, 8, …, corresponds to a cycle of notes C, E, G#, C, E, and G#…. This sequence is called a circle of thirds.

In much of classical and popular music, the circle of fifth intervals is employed. The 12-pointed star in the diagram traces the full circle advancing seven "hours" at a time from C through:

$$G, D, A, E, B, F\#, C\#, G\#, D\#, A\#, F, \text{ and back to C.}$$

This circle touches all twelve semitones. It is the well-known circle of fifths. From wikipedia:

> In practice, compositions rarely make use of the entire circle of fifths…. The usual practice is to derive the circle of fifths progression from the seven tones of the diatonic scale, rather from the full range of twelve tones present in the chromatic scale.

Diatonic scale? Ah! Now we get to explain "octave." The diatonic scale is the one most familiar to the ears of western culture denizen. You might sing it now: do, re, mi, fa, sol, la, ti, and then do again. Those "do" notes are eight steps apart in this diatonic scale.

A thorough treatment of this subject, including listening samples can be found **here**.

Tasks for you. Answers on page 4.
1. On the chromatic circle diagram, trace and list the notes in these circles: of fourths (starting C to F), of sixths (C to A), and of tritones (C to F#).

2. Is there a mathematical reason that some circles don't include all 12 notes?

3. Can you add example songs to the ones below for each interval?

Musical Intervals in Songs
(regarding the first two notes unless noted)

Selected Ascending:
Minor second: White Christmas; A Hard Day's Night
Major second: Happy Birthday after first note; Rudolph, the Red-Nose Reindeer
Minor third: Greensleeves; Georgia on My Mind
Major third: When Saints Go Marching In; Morning Has Broken
Fourth: Here Comes the Bride; Amazing Grace
Tritone: Maria two penultimate notes; The Simpson's Theme
Fifth: Twinkle, Twinkle Little Star; Scarborough Fair (both after the first note)
Augmented fifth: Joplin's The Entertainer after second note, repeated 3 times; Chopin's Waltz in C-sharp minor
Sixth: My Bonny Lies Over the Ocean; NBC chimes; My Way
Diminished seventh: Somewhere (There's a Place for Us)
Octave: Over the Rainbow

Selected Descending Intervals:
Minor third: Frosty the Snowman; Hey Jude
Major third: Swing Low Sweet Chariot; Beethoven's Fifth Symphony
Fourth: I've Been Working on the Railroad; All of Me
Fifth: The Way You Look Tonight; Flintstones Theme
Major sixth: Nobody Knows the Trouble I've Seen; The Music of the Night (fr Phantom)
Octave: Willow Weep for Me

Sources:
Kelly Leprohon
Musicca's Interval Song Chart

Answers to Your Tasks

1. The circle of fourths runs through all 12 tones in reverse order from the fifths. Why is that?
Circle of sixths: C, A, F#, D#, and C again.
Circle of tritones: C, F#, and C again.

2. The numbers 5 and 7 have no factors in common with 12, so the circles of fourths and fifths will touch every note before arriving at a multiple of 12. The sequence 0, 5, 10, 15, … has twelve values before reaching 60 (12 times 5, or five trips around the clock).

The tritone interval (6 half steps) returns to C after only two steps because 6 divides 12 twice.

Play On!!