Tessellating Hexagons: Escher's Theorem

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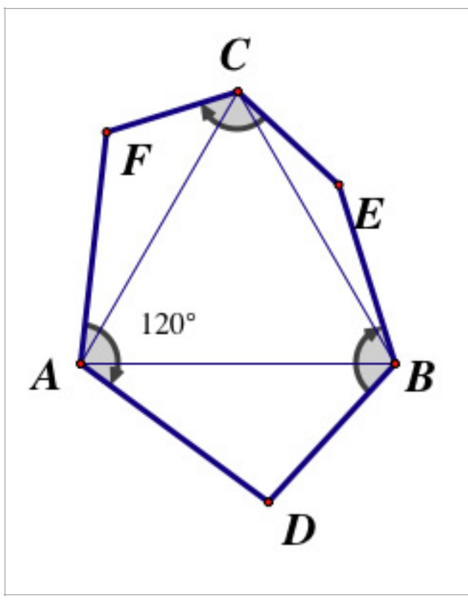
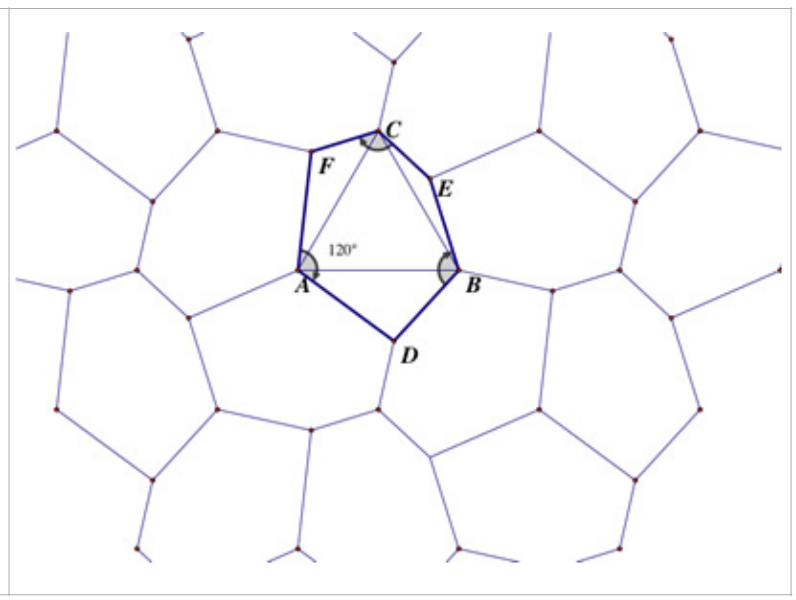
This article is intended to illuminate a theorem of Escher's that he was unable to prove formally. Using dynamic geometry software we show that the veracity of his idea is beyond question.

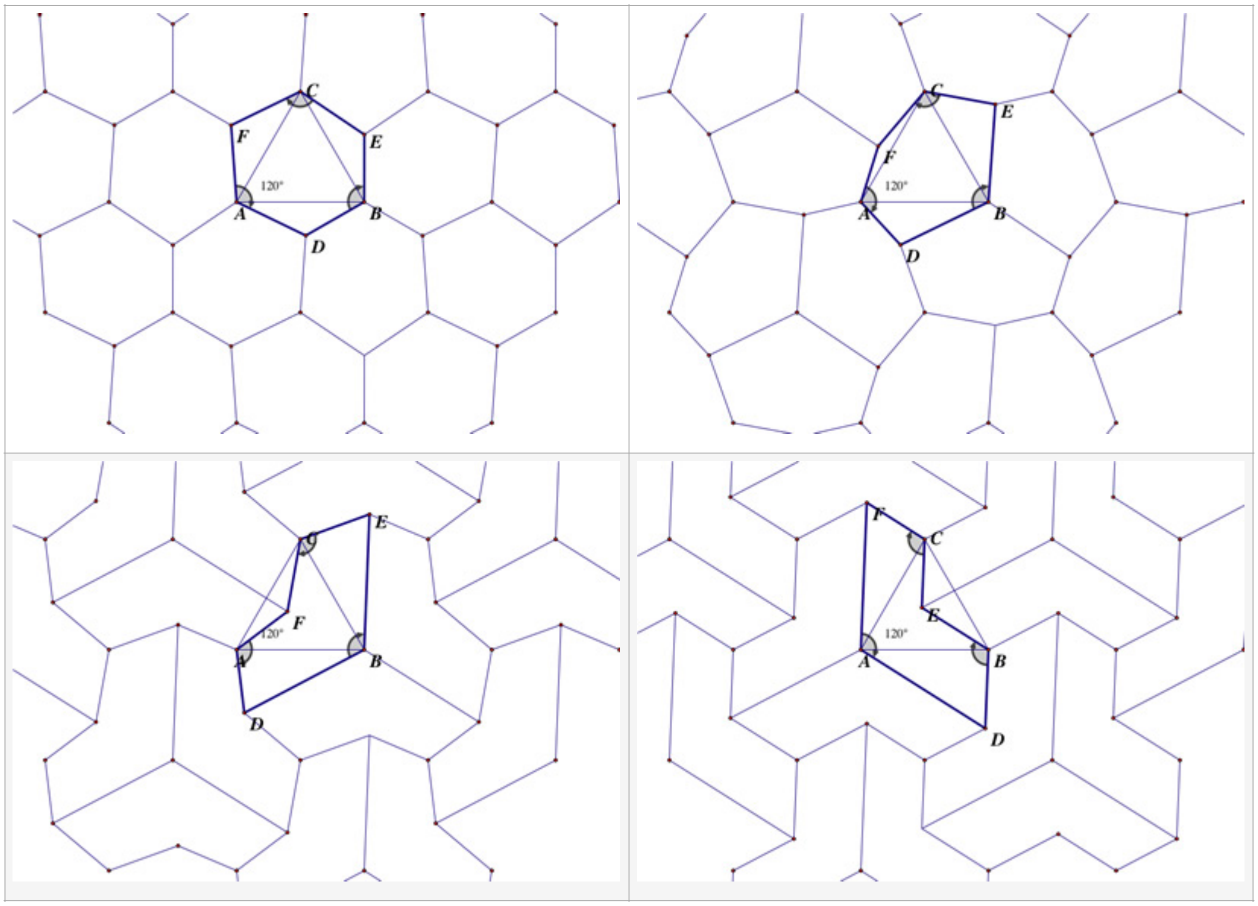
The Theorem: A hexagon with three 120-degree interior angles formed by congruent pairs of sides, and alternating between the other three angles, will tessellate the plane.

Demonstration: Begin with an equilateral triangle ABC. From point A travel an arbitrary distance in an arbitrary direction to locate a point F. This one independent point will determine the shape and size of the tessellating hexagon. Centering at A, swing an arc (clockwise in this case) 120 degrees from F to locate point D. From D swing 120 degrees about point B to locate E. Then FADBEC will be a hexagon that tiles the entire plane.





Observe first that the hexagon has three pairs of sides equal in length, the swinging steps, and next that three of six interior angles of the hexagon measure 120. The other three angles must sum to 360 degrees whatever their measures. It is fairly common knowledge that regular hexagons possessed of six 120-degree angles will fill the plane. That tiling is identified with the bee's honeycomb. Escher's theorem allows for an infinite, though not unrestricted, variety in the tiling design. On the computer, we dragged point F from the sketch [upper left] to alter the shape. Can you envision the transitions from one to another?



Source: Schattschneider, *Visions of Symmetry*.

Postscript: Keith Devlin [The Language of Mathematics p.217 and Plate 10] reports that in 1918 formal proofs were given for the tessellations of this and two other types of hexagons.

In my Escher Gallery I make extensive use of these hexagons and his technique for converting them into pentagons and equilateral triangles. See Wired, Violets, and Flowers in 5 and 6.